Lecture 8

$$u_t + f(u)_x = u_t + f'(u)u_x = 0 \qquad IC\ u(x,0) = g(x) \tag{1}$$

$f'(u) = u$ $Burgers$. One way to disallow loss of solution existence as a graph consider

$$\underbrace{u_t + f(u)_x = \epsilon u_{xx}}_{Viscosity} \qquad (\epsilon > 0) \tag{2}$$

Consider (1) as approximation of (2) with $\epsilon \to 0$

$$u_x \approx \frac{u}{l}$$

$$u_{xx} \approx \frac{u}{l^2}$$

Look at Fig 7.1

$$l \approx \epsilon \implies u_x \approx \epsilon u_{xx}$$

$$u_t = \epsilon u_{xx} - f'(u)u_x \tag{3}$$

Theorem 1 (Hopf'50, $f'(u) = u$). *(1) (2) has a unique bounded solution $u_\epsilon \in C^1(\mathbb{R} \times \{t > 0\})$*

$$s.t \qquad u_\epsilon(x,t) \to g(x) \qquad t \to 0$$

(2) $\exists D \subseteq \mathbb{R} \times \{t > 0\}$ s.t $\forall (x,t) \in D,\ u_\epsilon \to u_0(x,t),\ as\ \epsilon \to 0\ for\ some\ u_0(x,t)$. $|\mathbb{R} \times \mathbb{R}_{>0} \backslash D| = 0$.
(3) $u_0(x,t) \to g(t)$ as $t \to 0$ locally $uniformaly$.
(4) $u_0 \in L^\infty(\mathbb{R} \times \mathbb{R}_{>0})$ u_0 is called the viscosity solution of (1) is a weak solution of (1)

Proof. Will be discussed next term in PDE 2. One may read about this in Leveque Numerical methods for conservational laws. □

Weak Solutions

$\phi \in C_o^1(\mathbb{R}^2)$

$$0 = \int_0^\infty \int_R u_t\phi + f_x\phi. \qquad \int_R f_x\phi = f(u(x,t))\phi(x,t)\Big|_{x=x_1}^{x=x_2} - \int f\phi_x$$

$$\int_0^\infty u_t\phi = -u(x,0)\phi(x,0) - \int_0^\infty u\phi_t$$

$$\int_0^\infty \int_R u\phi_t + f(u)\phi_x = -\int_R u(x,0)\phi(x,0)dx \tag{4}$$

Definition 1. *A locally integrable function u is called a weak solution of (1) if (4) holds for any $\phi \in C_o^1(\mathbb{R}^2)$.*

Class notes by Ibrahim Al Balushi

Riemann Problem

We start with a piecewise constant data, and we see how it evolves. Fig 7.2

$$\frac{\partial}{\partial t} \underbrace{\int_{-\epsilon}^{\epsilon} u}_{u_-(\epsilon/2+st)+u_+(\epsilon/2-st)} = -f(u_+) + f(u_-)$$

$$(u_- - u_+)s = f(u_-) - f(u_+)$$

$$s = \frac{f(u_+ - f(u_-))}{u_+ - u_-} = \frac{[f(u)]}{[u]} \qquad Rankine - Hugomot\ condition.$$

Look Figs and their corresponding space time.

(Burger's)

$$s = \frac{1/2 - 0/2}{1 - 0} = 1/2$$

In the case of Rarefraction wave, we have none uniqueness. We require additional criterion : Entropy Conditions:

$$f'(u_-) > s > f'(u_+) \qquad \text{Characteristics must go into the shock}$$

Weak solutions which satisfy the entropy conditions called entropy solutions.

Numerical Methods for First order Eq.

-Characteristic Trancing: Solve ODE's on computer.
- Discretization. $\partial_t u = \frac{u(x,t+h)-u(x,t)}{h}$.